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On dynamic algebras

Siniša Crvenković and Rozália Sz. Madarász

Institute of Mathematics, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Serbia

Abstract

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Dynamic algebras are the Lindenbaum–Tarski algebras of dynamic logics. These algebras can be considered as Boolean algebras with some operators, indexed by the elements of some Kleene algebra. In this paper we prove that there are infinitely many finitely generated varieties of dynamic algebras having undecidable equational theories. All these varieties are generated by representable dynamic algebras.

1. Introduction

There are several algebraic structures which correspond to some notions from computer science. Such are Kleene and dynamic algebras. In the literature there are many algebras which are called Kleene algebras. We consider Kleene algebras which are obtained from the so-called Kleene relation algebras (without inversion). Kleene relation algebra, with some base U, is an algebra having the set of all binary relations on the set U as the carrier, and the fundamental operations are set-theoretical union, composition, and reflexive–transitive closure. Kleene algebra is an algebra (of appropriate type) that belongs to the variety generated by all Kleene relation algebras. Axiomatizations (equational or not) of the equational theory of Kleene relation algebras were obtained in [1, 3, 11, 13, 21] and with inversion in [7].

Because of the relationship between Kleene relation algebras and regular languages, it follows that the equational theory of Kleene algebras is decidable. The decidability of the equational theory of Kleene relation algebras with inversion has been established in [2]. In [6] it is proved that the word problem for the class of all Kleene algebras is unsolvable.

Correspondence to: S. Crvenković, Institute of Mathematics, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Serbia.

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Dynamic algebras ([10, 19]) are algebraic counterparts of propositional dynamic logic. Roughly speaking, dynamic logic is a classical propositional logic with some modal operators $\langle x\rangle$ associated with the elements x of a Kleene algebra. We can say that the corresponding algebraic structures, dynamic algebras, are Boolean algebras with normal unary operators which are indexed by the elements of a Kleene algebra. Although the equational theory of Kleene algebras is decidable, in this paper we prove that there are infinitely many finitely generated varieties of dynamic algebras having undecidable equational theories. All these varieties are generated by representable dynamic algebras.

2. Basic definitions

Definition 2.1. Let U be a set. A *Kleene relation algebra* is the algebra $\mathscr{K}(U)=(\mathscr{P}(U^2), \cup, \circ, {}^*)$, where $\cup$ is set-theoretic union, $\circ$ is composition of relations, $*$ is the reflexive–transitive closure of a relation, i.e.

$$\rho^* = \bigcap\{\sigma\subseteq U^2\colon \rho\subseteq\sigma \text{ and } \sigma \text{ is a reflexive and transitive relation on } U\}.$$

A *Kleene algebra* is an algebra $\mathscr{K}=(K, \vee, ;, {}^*)$ of type (2, 2, 1) that belongs to the variety generated by all Kleene relation algebras $\mathscr{K}(U)$.

Sometimes, Kleene relation algebras are called *inversion-free Kleene relation algebras*. Similarly, Kleene algebras are by some authors referred to as *inversion free-Kleene algebras.*

In the sequel, if $(B, \cdot, -)$ is a Boolean algebra, then $x+y$ abbreviates $-(-x\cdot -y)$, 0 abbreviates $x\cdot -x$, and $x\leqslant y$ abbreviates $x+y=y$.

Definition 2.2. Let $\mathscr{K}=(K, \vee, ;, {}^*)$ be a Kleene algebra. An algebra $\mathscr{D}=(B, \cdot, -, F_a$ $(a\in K))$ is a *dynamic $\mathscr{K}$-algebra* if it satisfies the following conditions:

(1) $(B, \cdot, -)$ is a Boolean algebra,
(2) $F_a(0)=0$,
(3) $F_a(x+y)=F_a(x)+F_a(y)$,
(4) $F_{a\vee b}(x)=F_a(x)+F_b(x)$,
(5) $F_{a;b}=F_aF_b(x)$,
(6) $x+F_aF_{a^*}(x)\leqslant F_{a^*}(x)$,
(7) $F_{a^*}(x)\leqslant x+F_{a^*}(-x\cdot F_a(x))$,

for all $a, b\in K$, $x, y\in B$.

Remark 2.3. Dynamic algebras can be considered as heterogeneous algebras. In this approach, a dynamic algebra $\mathscr{D}$ is a two-sorted algebra $\mathscr{D}=(\mathscr{A},\mathscr{B}, \diamond)$, where $\mathscr{B}$ is a Boolean algebra, $\mathscr{A}$ is an algebra of the type (2, 2, 1), $\diamond : A\times B\to B$, and $\mathscr{D}$ satisfies certain equations.

Definition 2.4. Let U be a set. Algebra $\mathscr{D}=(B, \cap, -, F_a\, (a\in K))$ is called a dynamic set algebra with base U if the following conditions hold:

(1) $(B, \cap, -)$ is a Boolean set algebra of some subsets of U, where $\cap$ is intersection and $-$ complementation,

(2) $(K, \cup, {}^*)$ is subalgebra of the Kleene relation algebra $\mathscr{K}(U)$,

(3) $F_a(X)=\{y\in U: (\exists x\in X)\langle x, y\rangle \in a\}$ for all a in K.

A dynamic algebra is called *representable* if it is isomorphic to some dynamic set algebra.

Of course, every dynamic set algebra is a dynamic algebra. Dynamic set algebras (actually, the two-sorted version) are called *Kripke structures* in [19].

3. The main result

Our aim is to transfer the undecidability result from Kleene algebras to dynamic algebras. We know (see [6]) that there is a Kleene algebra $\mathscr{K}_0$ with unsolvable word problem. The first idea could be to consider a dynamic algebra $\mathscr{D}=(B, F_a\,(a\in K_0))$, i.e. some dynamic algebra with operators indexed by the elements of this "bad" Kleene algebra $\mathscr{K}_0$. However, from the construction of the algebra $\mathscr{K}_0$ (see [6]) it is not clear how we should define the operators $F_a\,(a\in K_0)$ so that $\mathscr{D}$ becomes a dynamic algebra. Therefore, if we want to use the unsolvability of the word problem for Kleene algebras, it ought to be in a different way.

Definition 3.1. Let $\mathscr{S}$ be a semigroup with an identity. By $\mathscr{T}(\mathscr{S})$ we denote the so-called *semigroup of left translations of* $\mathscr{S}$, i.e. $\mathscr{T}(\mathscr{S})=(T(\mathscr{S}), \circ)$, where

$$T(\mathscr{S})=\{\lambda_s: \lambda_s \text{ is a left translation of } \mathscr{S},\ s\in S\},$$

$$\lambda_s: S\to S, \qquad \lambda_s(x)=s\cdot x,$$

and $\circ$ is the usual composition of functions. If a semigroup $\mathscr{S}$ has no identity element, then $\mathscr{T}(\mathscr{S})$ denotes the semigroup of left translations of the semigroup $\mathscr{S}^1$ (the semigroup obtained from $\mathscr{S}$ by adding an identity element).

It is obvious that for all semigroups $\mathscr{S}$ with an identity, $\mathscr{S}\cong\mathscr{T}(\mathscr{S})$. Note that we can consider the elements $\lambda\in T(\mathscr{S})$ as binary relations on the set S. So, $T(\mathscr{S})$ is a subset of the carrier of the Kleene relation algebra $\mathscr{K}(S)$.

Definition 3.2. Let $\mathscr{S}$ be a semigroup with an identity. By $\Psi(\mathscr{S})$ we denote the subalgebra of the Kleene relation algebra $\mathscr{K}(S)$ generated by the set $T(\mathscr{S})$. We define the dynamic set algebra $\mathscr{D}(\mathscr{S})$ to be $(\mathscr{P}(S), \cap, -, F_a\,(a\in\Psi(\mathscr{S})))$.

Let us recall some notions and notations connected with the word problem. Denote by $\mathscr{L}$ a first-order language which contains the symbol of identity relation $\approx$ and has no relation symbols. If G is a set of new symbols of constants ($\mathscr{L} \cap G = \emptyset$), then by $\mathscr{L}_G$ we denote the language $\mathscr{L} \cup G$. Usually, a symbol from G and its interpretation are denoted by the same letter. Let $\mathscr{A}$ be an algebra and $G \subseteq A$. Then by $\mathscr{A}_G$ we denote the algebra $(\mathscr{A}, x)_{x \in G}$. If R is a set of identities in $\mathscr{L}_G$ with no variables, then (G, R) is called a *presentation in* $\mathscr{L}_G$.

Definition 3.3. Let Θ be a set of identities of $\mathscr{L}$, V the variety defined by Θ and (G, R) a presentation in $\mathscr{L}_G$. For an algebra $\mathscr{A}$ in $\mathscr{L}$ we say that it is *presented by* (G, R) in V if the following hold:

(1) $\mathscr{A}$ is generated by G,
(2) $\mathscr{A}_G \models \Theta \cup R$,
(3) For any identity e in $\mathscr{L}_G$, with no variables, we have $\Theta \cup R \models e$ provided $\mathscr{A}_G \models e$.

The algebra $\mathscr{A}$ which is presented by (G, R) in V is denoted by $\mathscr{P}_V(G, R)$. For an algebra $\mathscr{B}$ we say that it is *finitely presented in* V if there are finite sets G and R such that $\mathscr{B}$ is presented by (G, R) in V. Note that the algebra presented by (G, R) in V is unique up to isomorphism.

Let Θ be a set of identities in the language $\mathscr{L}$, V the variety presented by Θ, and $\mathscr{A}$ the algebra finitely presented by (G, R) in V. The *word problem for* $\mathscr{A} = \mathscr{P}_V(G, R)$ *in* V inquires into the existence of an algorithm to determine, for any identity e in $\mathscr{L}_G$ with no variables, whether or not $\mathscr{A}_G \models e$. If such an algorithm exists, the word problem is *solvable*; otherwise it is *unsolvable*.

Definition 3.4. The *word problem for a variety* V asks if, for any finitely presented algebra $\mathscr{A}$ in V, there is an algorithm solving the word problem for $\mathscr{A}$ in V.

Theorem 3.5. *The word problem for the class of all Kleene algebras (also for Kleene algebras with inversion) is unsolvable.*

Proof. See [6]. □

Definition 3.6. The *semigroup of Cejtin* is the semigroup $\mathscr{C}$ presented by $(G(\mathscr{C}), R(\mathscr{C}))$, where

$$G(\mathscr{C}) = \{a, b, c, d, e\},$$

$$R(\mathscr{C}) = \{ac = ca,\ ad = da,\ bc = cb,\ bd = db,\ abac = adace.\ eca = ac,\ edb = be\}.$$

It is well known that the semigroup of Cejtin has unsolvable word problem.

Lemma 3.7. *There is a sequence* $\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_n, \ldots,$ *of finitely presented semigroups such that:*

(a) *all semigroups* $\mathscr{C}_i$ $(i \in N)$ *have unsolvable word problems,*
(b) $HSP(\mathscr{D}(\mathscr{C}_i)) \neq HSP(\mathscr{D}(\mathscr{C}_j))$ *for all* $i \neq j, i, j \in N$.

Proof. Let $\mathscr{C}$ be the Cejtin semigroup presented by (G, R), as in Definition 3.6, α a new symbol, and p_i the ith prime number, and SEM the variety of all semigroups. Let us define the required sequence of semigroups in the following way:

$$\mathscr{C}_0=\mathscr{C}, \quad \mathscr{C}_{i+1}=\mathscr{P}_{SEM}(G(\mathscr{C})\cup\{\alpha\}, \quad R(\mathscr{C})\cup\{\alpha^{p_i+1}=\alpha\}). \tag{1}$$

(a) It is obvious that all the semigroups defined by (1) are finitely presented. Also, it can easily be seen that if we had an algorithm for the solution of the word problem for any of the semigroups $\mathscr{C}_i$, we would be able to solve the word problem for the Cejtin semigroup, which is impossible.

(b) First of all, it is not hard to prove that the semigroup of Cejtin has no element of finite order, i.e. there is no $y\in C$ such that for some $n\in N$, $n\neq 1$, $y^n=y$. Also, because of (1), if $i<j$, then $\mathscr{C}_j$ has no element of order p_i.

Now, we will prove the following: if $\mathscr{S}$ is a semigroup presented by (G, R), and $\mathscr{D}(\mathscr{S})=(\mathscr{P}(S)\cap, -, F_\sigma\ (\sigma\in\Psi(S))$, then for every $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_k\in G$, the following holds:

$$\begin{aligned}&S_G\models a_1a_2\cdots a_n\approx b_1b_2\cdots b_k\\ &\quad\text{iff } \mathscr{D}(\mathscr{S})\models F_{\lambda_{a_1}}\cdots F_{\lambda_{a_n}}(X)\approx F_{\lambda_{b_1}}\cdots F_{\lambda_{b_k}}(X).\end{aligned} \tag{2}$$

(We can assume, without loss of generality, that $\mathscr{S}$ has an identity e.) We know that $(T(\mathscr{S}), \circ)$ is a semigroup which is isomorphic to $\mathscr{S}$. Let $G'=\{\lambda_g\colon g\in G\}$. We know that every element of $T(\mathscr{S})$ is of the form $\lambda_{a_1}\circ\lambda_{a_2}\circ\cdots\circ\lambda_{a_n}$, for some $a_1, \dots, a_n\in G$, so G' is the generating set for $\mathscr{T}(\mathscr{S})$. Also, if $a_1, \dots, a_n, b_1, \dots, b_k\in G$, and $\mathscr{S}_G\models a_1\dots a_n\approx b_1\cdots b_k$, then the corresponding equality $\lambda_{a_1}\circ\cdots\circ\lambda_{a_n}\approx\lambda_{b_1}\circ\cdots\circ\lambda_{b_k}$ holds in $\mathscr{T}(\mathscr{S})$. This is true in the opposite direction, too.

Now, we can consider in the Kleene relation algebra $\Psi(\mathscr{S})$ those elements, which are from $\mathscr{T}(\mathscr{S})$. If two expressions from $T(\mathscr{S})$, $\lambda_{a_1}\circ\cdots\circ\lambda_{a_n}$ and $\lambda_{b_1}\circ\cdots\circ\lambda_{b_k}$, are equal in $\mathscr{T}(\mathscr{S})$, they are equal in $\Psi(\mathscr{S})$ too and conversely. So, for any two words, $\lambda_{b_1}\circ\cdots\circ\lambda_{b_n}$ and $\lambda_{a_1}\circ\cdots\circ\lambda_{a_k}$ $(a_i, b_j\in G)$, it holds

$$\psi(\mathscr{S})_{G'}\models\lambda_{b_1}\circ\cdots\circ\lambda_{b_n}\approx\lambda_{a_1}\circ\cdots\circ\lambda_{a_k}$$

iff

$$\mathscr{T}(\mathscr{S})_{G'}\models\lambda_{b_1}\circ\cdots\circ\lambda_{b_n}\approx\lambda_{a_1}\circ\cdots\circ\lambda_{a_k}.$$

Starting from the Kleene algebra $\Psi(\mathscr{S})$ we construct the dynamic algebra

$$\mathscr{D}(\mathscr{S})=(\mathscr{P}(\mathscr{S}), \cap, F_\sigma\ (\sigma\in\Psi(S))),$$

where the operations $F_\sigma\colon\mathscr{P}(S)\to\mathscr{P}(S)$ are defined by

$$F_\sigma(X)=\{y\in S\colon(\exists x\in X)\langle x, y\rangle\in\sigma\}.$$

Since $G'\subseteq\Psi(S)$, then every $\lambda_g\in G'$, F_{λ_g} is an operation in $\mathscr{D}(\mathscr{S})$. Because of the axioms of dynamic algebras and because of the fact that $\mathscr{S}\cong\mathscr{T}(\mathscr{S})$, we have that for all $X\subseteq S$ it holds

$$F_{\lambda_{a_1}}\cdots F_{\lambda_{a_n}}(X)=F_{\lambda_{a_1\cdots a_n}}(X).$$

Now, it is obvious that if $\mathcal{S}_G \models a_1 \cdots a_n \approx b_1 \cdots b_k$ then $\mathcal{T}(\mathcal{S})_{G'} \models \lambda_{a_1} \circ \cdots \circ \lambda_{a_n} \approx \lambda_{b_1} \circ \cdots \circ \lambda_{b_k}$ and then

$$\mathcal{D}(\mathcal{S}) \models F_{\lambda_{a_1}} \cdots F_{\lambda_{a_n}}(X) \approx F_{\lambda_{b_1}} \cdots F_{\lambda_{b_n}}(X).$$

Conversely, if $\mathcal{S}_G \not\models a_1 \cdots a_n \approx b_1 \cdots b_k$, then $\mathcal{T}(\mathcal{S})_{G'} \not\models \lambda_{a_1 \cdots a_n} \approx \lambda_{b_1 \cdots b_k}$. So, the identity

$$F_{\lambda_{a_1}} \cdots F_{\lambda_{a_n}}(X) \approx F_{\lambda_{b_1}} \cdots F_{\lambda_{b_k}}(X).$$

does not hold in $\mathcal{D}(\mathcal{S})$ because if e is the identity element of $\mathcal{S}$, then

$$F_{\lambda_{a_1}} \cdots F_{\lambda_{a_n}}(\{e\}) = F_{\lambda_{a_1 \ldots a_n}}(\{e\}) = \{y \in S \colon \langle e, y \rangle \in \lambda_{a_1 \cdots a_n}\} = \{a_1 \cdots a_n\}.$$

$$F_{\lambda_{b_1}} \cdots F_{\lambda_{b_k}}(\{e\}) = F_{\lambda_{b_1 \ldots b_k}}(\{e\}) = \{y \in S \colon \langle e, y \rangle \in \lambda_{b_1 \cdots b_k}\} = \{b_1 \cdots b_k{}^n\}.$$

Hence, for every $a_1, \ldots, a_n, b_1, \ldots, b_k \in G$

$$\mathcal{S}_G \models a_1 \cdots a_n \approx b_1 \cdots b_k \text{ iff } \mathcal{D}(\mathcal{S}) \models F_{\lambda_{a_1}} \cdots F_{\lambda_{a_n}}(X) \approx F_{\lambda_{b_1}} \cdots F_{\lambda_{b_k}}(X).$$

So, we have proved (2). Now, in the case of semigroups $\mathcal{C}_i$ $(i \in N)$, we have that for $i<j$, it holds

$$\mathcal{C}_i \models \alpha^{p_i+1} \approx \alpha, \qquad \mathcal{C}_j \not\models \alpha^{p_i+1} \approx \alpha.$$

Because of (2), this means that $HSP(\mathcal{D}(\mathcal{C}_i)) \neq HSP(\mathcal{D}(\mathcal{C}_j))$. □

Theorem 3.8. *There are infinitely many finitely generated varieties of dynamic algebras, with countably many operations, having undecidable equational theories. All these varieties are generated by representable dynamic algebras.*

Proof. We proved in Lemma 3.7 that for all finitely presented semigroups $\mathcal{S} = \mathcal{P}_{SEM}(G, R)$ if holds

$$\mathcal{S}_G \models a_1 \cdots a_n \approx b_1 \cdots b_k \text{ iff } \mathcal{D}(\mathcal{S}) \models F_{\lambda_{a_1}} \cdots F_{\lambda_{a_n}}(X) \approx F_{\lambda_{b_1}} \cdots F_{\lambda_{b_k}}(X).$$

This means that if semigroup $\mathcal{S}$ has unsolvable word problem, the corresponding dynamic algebra $\mathcal{D}(\mathcal{S})$ has undecidable equational theory. Therefore, the variety $HSP(\mathcal{D}(\mathcal{S}))$ has also undecidable equational theory. Further on, because of Lemma 3.7, there is a sequence of finitely presented semigroups $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_n, \ldots,$ such that all members of this sequence have unsolvable word problems. Also, it is proved that these semigroups produce different varieties of dynamic algebras. So, we can conclude that there are infinitely many varieties of dynamic algebras with undecidable equational theories. □

4. Final remarks

In universal algebra several different problems of decidability are considered. Generally, if Σ is a set of formulas in some language $\mathcal{L}$, then Σ is said to be *decidable* if there is an algorithm which, for every formula φ in $\mathcal{L}$, decides whether $\varphi \in \Sigma$ or not.

So, if V is variety (or, more generally, a class of algebras), we can talk about decidability of the elementary theory, or equational theory or decidability of the theory of quasi-identities. The solvability of the word problem is somewhat different from the previous three decidability problems (see Definition 3.3). In literature, the word problem for a class of algebras is considered on two levels. More often, the word problem for the class V is defined as in Definition 3.4. This is the so-called *local word problem* or *the word problem on the second level* (WP II). As we can see, the local word problem asks whether, for any finitely presented algebra in V, there is an "individual" algorithm which solves the word problem. In the case of the *global word problem* or *the word problem on the first level* (WP I) we require the existence of a universal algorithm which solves the word problem for all finitely presented algebras in the given class V.

What is the relationship between these decidability problems? It is obvious that, for any class V, the decidability of elementary theory implies the decidability of the theory of quasi-identities, and this further implies the decidability of the equational theory. Also, it immediately follows that if the global word problem is solvable for V, then the local word problem for V is solvable too. In [8] it is emphasized that there is an essential difference between the two levels of the word problem. In [16] there are several examples of varieties which have solvable WP II but unsolvable WP I.

The fact that the problem of quasi-identities and WP I are equivalent is more or less in the "folklore" (see e.g. [15] or [14]). So, from Theorem 3.8 we can conclude the following:

Corollary 4.1. *There are infinitely many finitely generated varieties of dynamic algebras, with countably many operations, having unsolvable global word problems. All these varieties are generated by representable dynamic algebras.*

We can also note, that the undecidability of the equational theory, in general, does not imply the unsolvability of WP II. So, Theorem 3.8 does not give any information on the local word problems for varieties of dynamic algebras.

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